Research Statemen for Daniel Soskin

My research interests lie at the intersection of algebra and combinatorics. I am particularly interested in problems related to positivity phenomena in algebraic combinatorics, representation theory, cluster algebras. My studies are centered around inequalities arising from real algebraic geometry and tropical geometry which I aim to understand from combinatorial and computational complexity viewpoint. The list of my main projects includes:

1. Equalities and inequalites for balanced producs of Schur functions. In the joint work with I. Pak [47], we study Schur positivity of the difference of products of two skew Schur functions. Many special cases of this problem has been wide open for decades. Inspired by the theory of Temperley–Lieb immanants, we introduce balanced products, and give sufficient conditions for both Schur positivity and equality in this case. Applications include properties of the largest Littlewood–Richardson coefficients, a combinatorial interpretation of the defect of the Lam–Postnikov–Pylyavskyy inequality [36, Theorem 5], and proof of the Fomin–Fulton–Li–Poon conjecture [18, Conjecture 5.1] in the balanced case.

2. Bounded ratios for Lorentzian matrices. With D. Huang, J. Huh and B. Wang [12] we study multiplicative inequalities among entries of Lorentzian matrices through the notion of bounded ratios. These inequalities significantly generalize the classical Alexandrov–Fenchel and reverse Khovanskii–Teissier inequalities for mixed volumes. Based on Gromov's δ-hyperbolic metrics, we show that the cone of bounded ratios for Lorentzian matrices is the dual cone of the *cut cone*, a well-studied object in metric geometry, graph theory, and optimization [11]. In the case $n = 3$, we determine the optimal bounding constants on the entire cone, obtaining a closed formula with an entropy-like form. We conjecture that bounded ratios have a subtraction-free property which suggests a promising link between cluster algebras and Lorentzian polynomials and matroids.

Results on bounded ratios is a featured topic in Arbeitsgemeinschaften at Oberwolfach, Oct. 2025.

3. Description of multiplicative determinantal inequalities for totally positive matrices. The problem of description of multiplicative determinantal inequalities of minors was stated by S. Fallat and C. Johnson in [14] through the notion of bounded ratios. With M. Gekhtman [51], we show that the set of bounded ratios is finitely generated for any dimension and provide examples of new non-primitive generators which disprove Fallat–Gekhtman–Johnson characterization conjecture. With Z. Greenberg and M. Gekhtman [27] we found the *full list* of generators for $Gr(4, 8)$.

4. Bounded Laurent monomials in cluster variables on the positive loci. We extend the problem of description of multiplicative determinantal inequalities to all cluster variables. We show in [27] that for the full rank finite type cluster algebras the generators of these set are in bijection with mutable cluster variables and are of the form $\frac{y}{1+y}$, where y is a y-coordinate associated to a source in fully sources/sink orientation of the associated Dynkin diagram D in the exchange graph of the cluster algebras. In fact, the above set of generators corresponds to u-variables of cluster configuration algebra $\mathcal{U}_D$ defined in [3] by N. Arkani-Hamed, S. He, and T. Lam.

5. Hadamard products of dual Jacobi–Trudi matrices. With R. Angarone, J. S. Kim, J. Oh we study positivity properties of Hadamard products of Jacobi–Trudi matrices [48]. A. Sokal conjectured that Wagner's theorem [55] on total positivity of Hadamard product of triangular Toeplitz matrices can be strengthened to total monomial positivity for the Hadamard product of Jacobi–Trudi matrices. We show that Temperley–Lieb immanants are Schur positive for Hadamard products of Jacobi–Trudi matrices given by ribbon-like skew partitions. In particular, we affirm

Sokal's conjecture for minors given by ribbon-like skew partitions. Moreover, we provide a manifestly positive Schur expansion for the Hadamard product of Jacobi–Trudi matrices indexed by ribbons.

6. Majorizing monotonisity of Symmetrized Fischer's products for totally nonnegative matrices. A large class of determinantal inequalities is known to hold both for totally nonnegative matrices (TNN) and hermitian positive semidefinite matrices (HPSD). The classical examples are Hadamard-Fischer-Koteljanskii inequalities. W. Barrett and C. Johnson have proved that these inequalities can be generalized for the averages of certain products of minors of HPSD matrices [5]. We have shown in our joint work with M. Skandera that Barrett-Johnson inequalities hold for TNN matrices as well [50].

7. Plücker inequalities for weekly-separated coordinates in totally nonnegative Grassmanninan. In the join project with P. Vishwakarma [52] we have shown that the partial sums of Plücker relations form a family of nonnegative functions on the nonnegative part of Grassmannian for any two weekly-separated Plücker coordinates. This result connects several fundamental objects such as cluster algebra of a Grassmannian $Gr(k, n)$, long Plücker relations and Temperley-Lieb immanants.

1 Positivity phenomena and cluster algebras

Totally positive (nonnegative) matrix is matrix with real entries such that all its minors are positive (nonnegative). We denote these matrices as TP and TNN respectively. Total positivity arose initially in few different areas. It was studied by Gantmacher and Krein in oscillations of vibrating systems [25], by Schoenberg in applications to analysis of real roots of polynomials and spline functions [1]. TP and TNN matrices play important role in algebraic and enumerative combinatorics, integrable systems, probability, classical mechnics, and many other areas, see [2], [26], [15] and references therein. Lusztig extended the notion of total positivity to reductive Lie groups G [43], where totally nonnegative part $G^{\geqslant 0}$ of G is a semialgebraic subset of G generated by Chevalley generators. $G^{\geqslant 0}$ is subset of G where all elements of dual canonical basis are nonnegative [42]. This concept could be generalized even further to varieties V. Totally nonnegative subvariety is defined as a subset of V where certain regular functions on V have nonnegative values [30], [6], [19]. Lusztig proved that specialization of elements of the dual canonical basis in representation theory of quantum groups at q=1 are totally nonnegative polynomials. Thus, it is important to investigate classes of functions on varieties that attain positive values on positive subvarieties. I will discuss several sources of such functions later.

A natural question is to describe the minimal subsets of minors which are sufficient to test for positivity in order to guarantee that the matrix is TP [20]. This question was one of the main motivations for the cluster algebras theory initially introduced by Fomin and Zelevinsky [21]. It turns out that for a $n \times n$ matrix there are multiple *clusters* of n^2 minors such that positivity of minors in any fixed cluster ascertains that the matrix is TP. The key role in total positivity test is played by the *Laurent phenomenon* [22], that is every minor of the matrix can be expressed as a *subtraction-free* Laurent polynomial in minors of any fixed cluster. Subtraction-free expressions are expressions involving no negative signs. Since then theory of cluster algebras found application in a plethora of different branches of mathematics such as representation theory of quivers and finite-dimensional algebras [9], [38], discrete dynamical systems based on rational recurrences and Y -systems in the thermodynamic Bethe Ansatz [23], [29],[28], higher Teichmüller spaces [37], [17], Poisson geometry and theory of integrable systems [30].

Multiplictive determinantal inequalities through the notion of bounded ratios. A large list of inequalities hold for TNN matrices and HPSD matrices. Hadamard [31] showed that for A HPSD we have

$$\det(A) \leqslant a_{1,1} \cdots a_{n,n}, \tag{1.1}$$

and Koteljanskii [34], [35] showed that this holds for A TNN as well. Marcus [44] proved a permanental analog

$$\operatorname{per}(A) \geqslant a_{1,1} \cdots a_{n,n} \tag{1.2}$$

for A HPSD, and this analog clearly holds for A TNN as well. Fischer [16] strengthened (1.1) by showing that for all $I \subseteq [n]$ we have

$$\det(A) \leqslant \det(A_{I,I}) \det(A_{I^c,I^c}), \tag{1.3}$$

and Ky Fan showed that this holds for A TNN as well (unpublished; see [10]). Lieb [40] proved a permanental analog

$$\operatorname{per}(A) \geqslant \operatorname{per}(A_{I,I}) \operatorname{per}(A_{I^c,I^c}), \tag{1.4}$$

for A HPSD, and this analog holds for A TNN. Koteljanskii [34], [35] strengthened (1.3) further by proving that for all $I, J \subseteq [n]$ we have

$$\det(A_{I\cup J,I\cup J}) \det(A_{I\cap J,I\cap J}) \leqslant \det(A_{I,I}) \det(A_{J,J}) \tag{1.5}$$

for A belonging to a class of matrices including HPSD and TNN matrices.

Hadamard–Fischer–Koteljanskii inequalities are examples of multiplicative determinantal inequalities. *We explore generalizations of these classical results in three contexts: TP Grassmannians, finite type cluster algebras and Lorentzian matrices.*

S. Fallat and C. Johnson formulated a question on description of multiplictive determinantal inequalities through the notion of bounded ratios in [14]. Let $I, I' \subseteq \{1, 2, \ldots, n\}$ with $|I| = |I'|$, we denote the minor of A with row set I and column set I' as $\Delta_{I,I'}(A) := \det A(I|I')$.

Problem 1.1. Describe ratios R of products of minors bounded on the locus of TP elements in GL_n, where R is of the form

$$R = \Delta_{I_1,I'_1}(A)\Delta_{I_2,I'_2}(A)...\Delta_{I_p,I'_p}(A)/\Delta_{J_1,J'_1}(A)\Delta_{J_2,J'_2}(A)...\Delta_{J_q,J'_q}(A) \tag{1.6}$$

Over the following twenty years some partial results have been obtained for several classes of inequalities and small dimensions [14], [13], [7], [49].

With M. Gekhtman we used the standard embedding of $M_{n\times n}$ into Grassmannian $Gr(n, 2n)$ to reformulate the above problem in terms of positive Plücker coordinates. We study the cone of bounded ratios via planar network parameterization of TP Grassmannian $Gr^+(n, 2n)$ and show that the cone of bounded ratios is polyhedral.

Theorem 1.1 (Gekhtman–S. [51])**.** For any n the cone of bounded ratios is finitely generated.

Moreover, we found new multiplicative inequalities which are not implied by quadratic inequalities that follow from *3-term Plücker relations*. By this we have disproved Fallat–Gekhtman–Johnson characterization conjecture stated in [13].

Bounded ratios play a role not only in the study of classical total positivity but also have applications in theoretical physics (e.g. in convergence of Koba–Nielsen string integrals [4]) and connections to tropical geometry as our recent result with N.Early and M.Gekhtman illustrates.

Theorem 1.2 (Early–Gekhtman–S. [46])**.** *We proved that the cone of bounded ratios is dual to the cone spanned by the rays of positive tropical Grassmannian $Trop^+Gr(k, n)$ defined in [53].*

With M. Gekhtman and Z. Greenberg [27] we provided the *full list* of extreme ratios of the bounded cone for $Gr^+(4,8)$. We discuss a generalization of the Problem 1.1 to a problem of bounded ratios in all cluster variables of the corresponding cluster algebra.

Problem 1.3. Describe ratios of products of cluster variables bounded as a real-valued function on the totally positive locus.

In [27] we characterize the cone of bounded ratios in Problem 1.3 for full rank cluster algebras of finite type. Generators of this cone are in bijection with mutable cluster variables, and they correspond to inequalities obtained from exchange relations associated to vertices of the Dynkin type quivers in the exchange graph of the cluster algebra. Moreover, generators of the cone of bounded ratios correspond to u-variables and satisfy u-equations defined by N. Arkani-Hamed, S. He, and T. Lam in [3].

Theorem 1.4 (Greenberg–Gekhtman–S. [27])**.** *Let $\mathcal{D}$ be a full rank finite type cluster algebra associated to a Dynkin diagram D. Then*

1. *The generators of the cone of bounded ratios are of the form $\frac{\prod_{\gamma\to\omega} x_\omega}{x_\gamma x'_\gamma}$ where x_γ is the variable at the source of a Dynkin type bipartite quiver that mutates to x'_γ.*

2. *The generators of the bounded cone correspond to u-variables of the corresponding cluster configuration algebra and satisfy u-equations $u_i + \prod_{j\neq i} u_j^{\alpha(i,j)} = 1$ (see [3] for definition).*

3. *Every bounded ratio in cluster variables of $\mathcal{D}$ is bounded by* 1*.*

Recently P. Brändén and J. Huh introduced Lorentzian polynomials [8], which unlocked proofs of long standing conjectures such as Mason's ultra-log-concavity, and forged deep connections between algebraic geometry, convex geometry, and combinatorics. In particular, Lorentzian bilinear forms are essential in space-time geometry and relativity. The celebrated Alexandrov–Fenchel inequality $v_{ii}v_{jj} \leqslant v_{ij}^2$ states that the mixed volume, restricted to two variables, has Lorentzian signature. Coefficients of Lorentzian bilinear forms are known to satisfy a special case of the reverse Khovanskii–Teissier inequalities $a_{ii}a_{kj} \leqslant 2a_{ik}a_{ij}$ [39], as well as their variations appearing in Brunn–Minkowski theory [24]. The Castelnuovo Severi inequality [45] used in Weil's proof of Riemann hypothesis for curves can also be viewed as an example of such inequality. Thus, it is natural to extend studies of determinantal inequalities from TP and HPSD matrices to Lorentzian matrices, which are symmetric matrices with nonnegative real entries and have at most one positive eigenvalue.

With D. Huang, J. Huh and B. Wang [12] we describe the set of multiplicative inequalities in entries of a Lorentzian matrices through the notion of bounded ratios. The key idea is that the set of entry-wise logarithms of Lorentzian matrices lies between two sets of matrices parametrized by Gromov's δ-hyperbolic metrics on n points (for $\delta = 0$ and $\delta = \log 2$). The Gromov tree approximation theorem implies that all three sets above are within finite Hausdorff distance, and thus all three sets have the same cones of bounded ratios associated to them.

Theorem 1.5 (Huang–Huh–S.–Wang [12])**.** *The cone of bounded ratios in entries of Lorentzian matrices of order n is dual to the cut cone Cut_n.*

Cut cone is a well-studied object [11]. In particular, its extreme rays suggest new inequalities stronger than Alexandrov–Fenchel inequalities. For matrix of order at least 5 we show that

$$a_{12}a_{13}a_{23}a_{44}a_{45}a_{55} \leqslant 4a_{13}a_{14}a_{15}a_{23}a_{24}a_{25}$$

However, a result of R. Karp and C. Papadimitriou shows that unless NP=coNP, there is no computationally tractable description of all facets of the cut cone [33]. Therefore, the theorem above is the best description of the cone of bounded ratios one can hope for.

2 Positivity phenomena and Immanants

Immanants are closely related to positivity phenomena and often suggest a surprising approach to open problems on positivity. Following [41], [54], for $f : \mathfrak{S}_n \to \mathbb{C}$ we define the *f-immanant* to be the polynomial

$$\mathrm{Imm}_f(x) := \sum_{w \in \mathfrak{S}_n} f(w) x_{1,w_1} \cdots x_{n,w_n} \in \mathbb{C}[x]. \tag{2.1}$$

Different choices of the function $f : \mathfrak{S}_n \to \mathbb{C}$ define various families of immanants. We are particularly interested in Temperley–Lieb immanants which are known to be elements of the dual canonical basis, see [32] for a detailed exposition. For example, Temperley–Lieb immanants are positive when evaluated on a TP matrix, and are Schur positive when evaluated on a Jacobi–Trudi matrix.

With I. Pak we study Schur positivity of the difference of products of two skew Schur functions

$$s_{\lambda/\nu} \cdot s_{\mu/\tau} - s_{\alpha/\gamma} \cdot s_{\beta/\delta} \geqslant_s 0. \tag{2.2}$$

Schur positivity of an expression of this form is equivalent to some inequalities between Littlewood-Richardson coefficients. Characterizing such inequalities is a generalization of the Klyachko problem on nonzero Littlewood-Richardson coefficients. Inequalities on Littlewood-Richardson coefficients have many applications, for example, to understanding of containment of tensor products of irreducible sl_n-modules $V_\lambda \otimes V_\mu$. Inspired by the theory of Temperley–Lieb immanants, we introduced *balanced products*. We say that pairs $\{\lambda, \mu\}$ and $\{\alpha, \beta\}$ are balanced if $\lambda^\triangleleft \cup \mu^\triangleleft = \alpha^\triangleleft \cup \beta^\triangleleft$, where $\lambda^\triangleleft = \lambda + \rho_n$ and ρ_n is a staircase partition, and $\mu^\triangleleft$, $\alpha^\triangleleft$, $\beta^\triangleleft$ are defined similarly. In this case, we give sufficient conditions for both Schur positivity and equality which can be verified in polynomial time with respect to the input size. Based on the obtained inequalities, we strengthen the result of I. Pak, G. Panova and D. Yeliussizov on large Littlewood-Richardson coefficients.

Theorem 2.1 (Pak–S. [47])**.** *Let $\mathcal{P}_n$ be the set of partitions with n nonnegative parts. For every three partitions $\lambda, \mu, \nu \in \mathcal{P}_n$ there exist $\alpha, \beta \in \mathcal{P}_n$ such that $c^\nu_{\lambda\mu} \leqslant c^\nu_{\alpha\beta}$, $\beta \subseteq \alpha$ and α/β is a disjoint union of ribbons.*

We study the defect of the Lam–Postnikov–Pylyavskyy inequality $s_{\lambda\vee\mu} \cdot s_{\lambda\wedge\mu} - s_\lambda \cdot s_\mu \geqslant_s 0$, where partitions $\lambda \vee \mu$ and $\lambda \wedge \mu$ are given as the union and intersection, respectively, of the corresponding Young diagrams, and answer one of the open questions posed at OPAC 2022.

Theorem 2.2 (Pak–S. [47])**.** *$c^\nu_{\lambda\vee\mu\ \lambda\wedge\mu} - c^\nu_{\lambda\mu}$ is in the complexity class $\#\mathbf{P}$.*

Following [18], for every two partitions $\lambda, \mu \in \mathcal{P}_n$, we define $\lambda^*, \mu^* \in \mathcal{P}_n$ as follows:

$$\lambda^*_k = \lambda_k - k + \#\{l |\ \mu_l - l \geqslant \lambda_k - k\}, \qquad \mu^*_l = \mu_l - l + 1 + \#\{k |\ \lambda_k - k > \mu_l - l\}.$$

Fomin–Fulton–Li–Poon Conjecture [18, Conjecture 5.1] states that $s^*_\lambda s^*_\mu - s_\lambda s_\mu \geqslant_s 0$. We prove this conjecture assuming that pairs of partitions $\{\lambda, \mu\}$ and $\{\lambda^*, \mu^*\}$ are balanced, which provides a wide family of new inequalities.

Theorem 2.3 (Pak–S. [47])**.** *Let $\lambda, \mu \in \mathcal{P}_n$ such that $\lambda^\triangleleft \cup \mu^\triangleleft = (\lambda^*)^\triangleleft \cup (\mu^*)^\triangleleft$, then $s^*_\lambda s^*_\mu - s_\lambda s_\mu \geqslant_s 0$.*

Another surprising application of Temperley–Lieb immanants appeared in the studies of positivity properties of Hadamard product of Jacobi–Trudi matrices. Recall that for a skew-shape λ/μ Jacobi–Trudi matrix is defined as follows

$$E_{\lambda/\mu}(\mathbf{x}) := \left(e_{\lambda_i - \mu_j - i + j}(\mathbf{x})\right)^{\ell(\lambda)}_{i,j=0}.$$

With R. Angarone, J. S. Kim and J. Oh we show that Temperley–Lieb immanants evaluated on Hadamard product of Jacobi–Trudi matrices given by ribbon-like skew-shapes are Schur positive.

Theorem 2.4 (Angarone–Kim–Oh–S. [48]). *Suppose $\lambda^{(1)}/\mu^{(1)}, \ldots, \lambda^{(k)}/\mu^{(k)}$ is a collection of skew partitions not containing a 3×2 block of cells. Then for any Temperley-Lieb immanant Imm_τ, the multi-symmetric function*

$$\mathrm{Imm}_\tau \left(E_{\lambda^{(1)}/\mu^{(1)}} \left(\mathbf{x^{(1)}} \right) * \cdots * E_{\lambda^{(k)}/\mu^{(k)}} \left(\mathbf{x^{(k)}} \right) \right)$$

is Schur positive.

A. Sokal conjectured that Wagner's theorem [55] on total positivity of Hadamard product of triangular Toeplitz matrices can be strengthened to total monomial positivity for the Hadamard product of Jacobi–Trudi matrices. As a corollary of the Theorem 2.4, we affirm Sokal's conjecture for minors given by 3×2-avoiding skew partitions.

Theorem 2.5 (Angarone–Kim–Oh–S. [48]). *Suppose λ/μ is a skew partition not containing a 3×2 block of cells. Then*

$$\det \left(E_{\lambda/\mu} \left(\mathbf{x} \right) * E_{\lambda/\mu} \left(\mathbf{y} \right) \right)$$

is Schur positive.

Moreover, we provide a manifestly positive Schur expansion for Temperley-Lieb immanants evaluated on Hadamard product of Jacobi–Trudi matrices indexed by ribbons. For a ribbon R we define $d(I,R) = \left\{ \sum_{j=1}^{i} (\lambda_j - \mu_j) : i \in I \right\}$ for any $I \subseteq [n-1]$.

Theorem 2.6 (Angarone–Kim–Oh–S. [48]). *For any ribbons $R^{(1)}, \ldots, R^{(k)}$, each with n rows, and any $I \subseteq [n-1]$, we have*

$$\begin{aligned} &\mathrm{Imm}_{\tau_I} \left(E_{R^{(1)}} \left(\mathbf{x^{(1)}} \right) * \cdots * E_{R^{(k)}} \left(\mathbf{x^{(k)}} \right) \right) \\ &= \sum_{\nu_1 \vdash m_1, \ldots, \nu_k \vdash m_k} \left(\sum_{I_1 \cup \cdots \cup I_k = I} f^{\nu_1}(d(I_1, R^{(1)})^c) \cdots f^{\nu_k}(d(I_k, R^{(k)})^c) \right) s_{\nu_1} \left(\mathbf{x}^{(1)} \right) \cdots s_{\nu_k} \left(\mathbf{x}^{(k)} \right). \end{aligned}$$

Here m_i is the size of $R^{(i)}$, and $f^{\nu_i}(A)$ denotes the number of standard Young tableaux with shape ν_i and descent set A.

A natural generalization of the classical Hadamard–Fischer–Koteljanskyy results is the majorizing monotonicity of symmetrized Fischer's products, which W. Barrett and C. Johnson have proved [5] for (real) PSD matrices. Partition μ *majorize* partition λ of n, $\lambda \preceq \mu$ if and only if $\lambda_1 + \cdots + \lambda_i \leqslant \mu_1 + \cdots + \mu_i$ for all i. With M. Skandera we show that these inequalities hold for TNN matrices as well. The key idea is that we express the defect of the inequality as a positive expression in Temperley–Lieb immanants.

Theorem 2.1 (Skandera–S. [50]). If $\lambda \preceq \mu$ then for any TNN matrix A the following inequality holds

$$\lambda_1! \cdots \lambda_r! \sum_{(I_1,\ldots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}) \ \geqslant\ \mu_1! \cdots \mu_s! \sum_{(J_1,\ldots,J_s)} \det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s}), \tag{2.3}$$

where sums are over sequences of disjoint subsets of $\{1, \ldots, n\}$ satisfying $|I_k| = \lambda_k$, $|J_k| = \mu_k$.

With P.Vishwakarma [52] we prove that the partial sums of Plücker relations for two weakly-separated coordinates form a family of positive functions on $Gr^+(k,n)$. Let I, I' be two k-element subsets of $[n]$, and let $I \setminus I' := [i_1, i_2, ..., i_m]$ and $I' \setminus I := [j_1, j_2, ..., j_m]$ be two ordered intervals of elements. Then I, I' are weakly-separated if $i_1 < \ldots < i_a < j_1 < \ldots < j_m < i_{a+1} < \ldots < i_m$ (or the same inequalities with i's and j's switched). The following theorem shows that partial sums of long Plücker relations oscillate around 0 for TNN matrices.

Theorem 2.2 (S.–Vishwakarma [52]). Let $I, I', [i_1, i_2, ..., i_m], [j_1, j_2, ..., j_m]$ be defined as above. Suppose $3 \leqslant m$, fix r such that $1 \leqslant r \leqslant m$ and let

$$I_s = (I \setminus \{i_r\}) \cup \{j_s\}, \quad I'_s = (I' \setminus \{j_s\}) \cup \{i_r\}, \quad \text{if } 1 \leqslant s \leqslant m.$$

Then for the following inequalities hold for $Gr^+(k, n)$

$$\sum_{s=1}^{l} (-1)^{s+l} \mathcal{P}_{I_s} \mathcal{P}_{I'_s} \geqslant 0, \quad l \in [1, m-r], \tag{2.4}$$

$$(-1)^{m-r+l} \mathcal{P}_I \mathcal{P}_{I'} + \sum_{s=1}^{l} (-1)^{s+l} \mathcal{P}_{I_s} \mathcal{P}_{I'_s} \geqslant 0, \quad l \in [m-r+1, m].$$

Future research directions.

1. We conjecture that bounded ratios for Lorentzian matrices satisfy subtraction-free property in terms of positive parametrization of Lorentzian matrices of rank at most 2. Conjecturally, bounded ratios in Plücker coordinates of $Gr^+(k, n)$ are subtraction-free in terms of positive parameters of the Fomin-Zelevinskyy network parametrization of $Gr(k, n)$. Bounded ratios in cluster variables of full rank finite type are known to be subtraction-free. These observations suggest a promising link between cluster algebras and Lorentzian polynomials and matroid theory which I am eager to explore.

2. I am interested to complete the characterization of partitions λ, μ, ν, τ and $\alpha, \beta, \gamma, \delta$ such that the following inequality holds $s_{\lambda/\nu} \cdot s_{\mu/\tau} - s_{\alpha/\gamma} \cdot s_{\beta/\delta} \geqslant_s 0$.

3. I am interested to complete the characterization of partitions λ, μ, ν, τ such that $s_{\lambda/\nu} = s_{\mu/\tau}$.

4. For a partition $\lambda \subset (n-k)^k$, the Chern-Schwartz-MacPherson (CSM) $\mathrm{csm}(X^\circ_\lambda)$ and the Kazhdan-Lusztig (KL) class $KL(X_\lambda)$ are non-homogeneous classes in $H^*(\mathrm{Gr}(k, n))$. I am interested to prove the following log-concave inequalities (verified for small partitions on a computer): let λ, μ be partitions inside the $k \times (n-k)$ rectangle. Then

$$\mathrm{csm}(X^\circ_{\lambda\cup\mu}) \cdot \mathrm{csm}(X^\circ_{\lambda\cap\mu}) - \mathrm{csm}(X^\circ_\lambda) \cdot \mathrm{csm}(X^\circ_\mu) \geq 0,$$

   and

$$KL(X_{\lambda\cup\mu}) \cdot KL(X_{\lambda\cap\mu}) - KL(X_\lambda) \cdot KL(X_\mu) \geq 0.$$

   Here ≥ 0 means that the class is a nonnegative combination of Schubert classes.

5. I am interested to find a counterexample or a prove to Sokal's conjecture mentioned before, as well as study positivity properties of classical, monomial, and other immanants related to various generalizations of Jacobi-Trudi matrices (such as Hamel–Goulden matrices).

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